Towards an Operational Semantics of the Simulation Engine of Simulink

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Motivation

Simulink and its industrial uses

Context

Simulink is a `de facto` standard in the industry to design embedded safety critical applications.
e.g. it is used to design almost all the **drive-by-wire** systems.

Components of a drive-by-wire systems

- A **physical mechanism** we want to control, e.g. a suspension.
- A **software** which implement the controller.

$\Rightarrow$ we have to deal with **hybrid systems**.

Current validation method

The main method to validate the design of embedded systems is based on **simulation activity**.

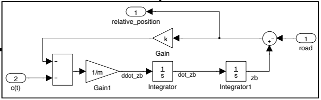
Classical design and validation methodology

Identification

Mathematical model

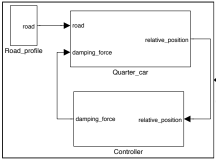
$$m\ddot{x} + c\dot{x} + kx = u$$

Computer description



Theoretical and experimental studies

Controller definition



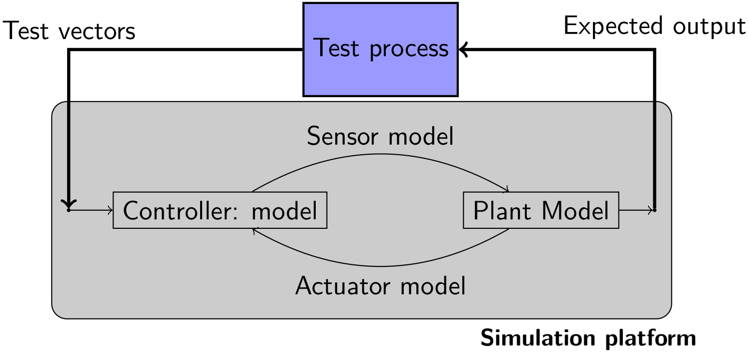
Remarks

- **Mathematical models** are an approximated descriptions of **physical systems**.
- The **computer descriptions** are studied with numerical tools.

Classical design and validation methodology

MIL: Model in the loop

Definition of a mathematical model of the `plant` and the `controller`.



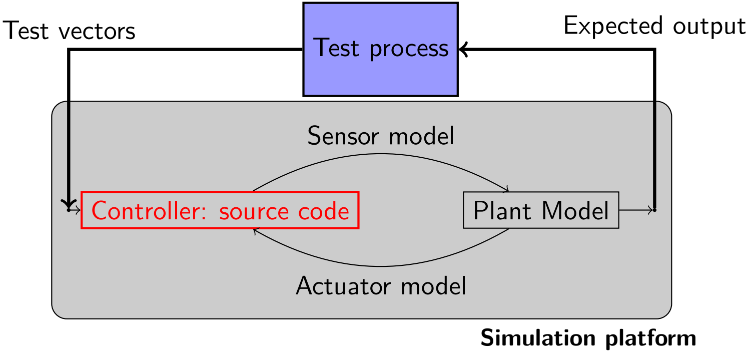
Aim of the testing activity

- Does the controller fulfil the specification?
- The set of tests will be used as "an oracle" in the next steps.

Classical design and validation methodology

SIL: Software in the loop

Implementation of the `controller` in a target language.



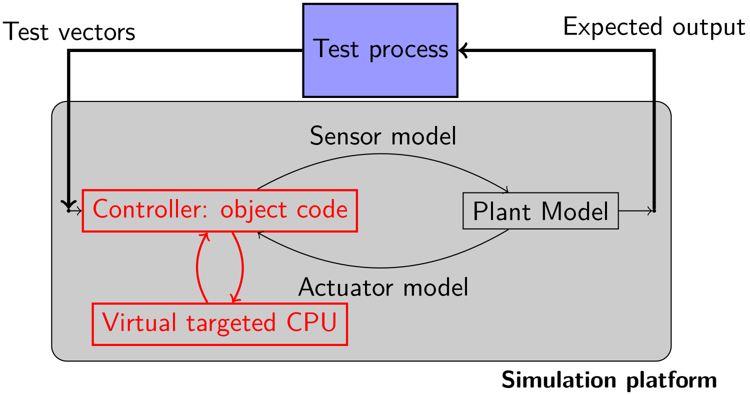
Aim of the testing activity

- Do the hand-written or generated code still fulfil the specification?

Classical design and validation methodology

PIL: Processor in the loop

Compilation of the controller and execute it on a virtual processor.



Aim of the testing activity

- Does the low-level implementation still fulfil the specification?

Considering Simulink as a language

Simulink: a graphical language

It allows to describe, as block-diagrams, a mixing of:

- **Ordinary differential equations** (ODE).
- Finite difference equations: **single-rate** or **multi-rate** period.
- Conditional executed equations: **enabled** or **triggered**.

Simulink: several parametrized semantics

- the definition of a **numerical solver** used to solve ODEs;
- the detection of events: **zero-crossing**.

So, it is a **numerical approximation** of the **mathematical behavior**.

What we should keep in mind!

The **designer** only refers to the **Simulink's semantics** (i.e. graphical output) to **validate** the development steps!

Formal verification of Simulink

Fact: simulations will never be complete and cannot bring strong guarantee on the designed software.

Main question: How can we help designer to be more confident in its designed systems with formal methods?

Formal verification of Simulink: **Our Goal**

- To compute invariant properties on the `software` **taking into account** `a model of the plant`.
- **Our approach:** abstract interpretation-based static analysis.

Formal verification of Simulink: **Constraint**

- Formal verification methods should be adaptable for the design process in order to be more easily adopted by the end-user.

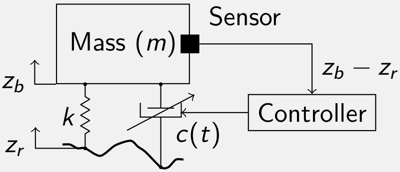
Formal verification of Simulink: **Challenge**

- Understand the **simulation engine** of Simulink.

Simulink Language

A simple example: mathematical models

A semi-active suspension of a quarter-car model



- $m = 250$ kg
- $k = 20000$ N/m
- $c_{\max} = 16000$ N/m/s
- $c_{\min} = 0$

Mathematical model of the mechanical system

$$\ddot{z}_b = -\frac{1}{m}\Big(k(z_b - z_r) + c(t)\Big) \ .$$

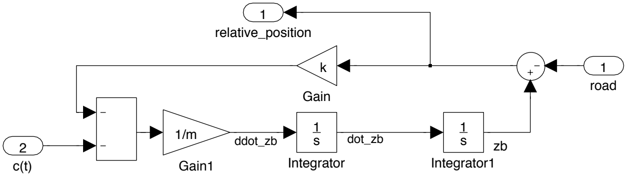
Mathematical model of the controller

$$c(t) = \begin{cases} -c_{\max}(z_b - z_r) & \text{if } (z_b - z_r)(\dot{z}_b - \dot{z}_r) < 0 \\ c_{\min} & \text{if } (z_b - z_r)(\dot{z}_b - \dot{z}_r) \geq 0 \end{cases} \ .$$

A simple example: Simulink implementation – 1

Mathematical model of the mechanical system

$$\ddot{z}_b = -\frac{1}{m}\Big(k(z_b - z_r) + c(t)\Big) \quad .$$



Integrator block

Associated to a first order dynamic system:

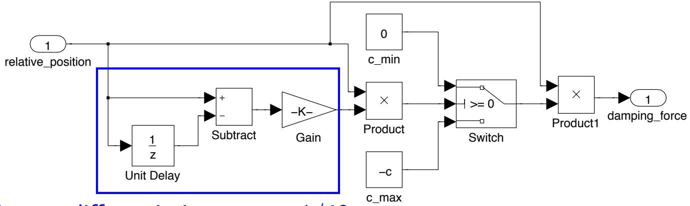
$$\begin{cases} \dot{x}(t) = \mathrm{input}(t) \\ \mathrm{output}(t) = x(t) \end{cases} \quad \text{with} \quad x(0) = x_0$$

A simple example: Simulink implementation – 2

Mathematical model of the controller

$$c(t) = \begin{cases} -c_{\max}(z_b - z_r) & \text{if } (z_b - z_r)(\dot{z}_b - \dot{z}_r) < 0 \\ c_{\min} & \text{if } (z_b - z_r)(\dot{z}_b - \dot{z}_r) \geq 0 \end{cases}.$$

Implementation: $(\dot{z}_b - \dot{z}_r)$ is given by differentiating the sensor output.



Discrete differentiation at rate $1/40$ sec.

Closed-loop system

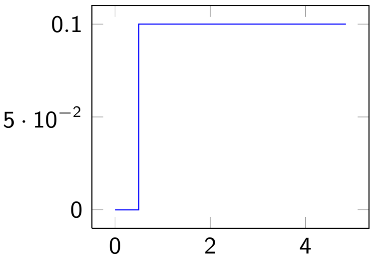
- Connect the output of the plant to the input of the controller.
- Connect the output of the controller to the input of the plant.

A simple example: simulation results

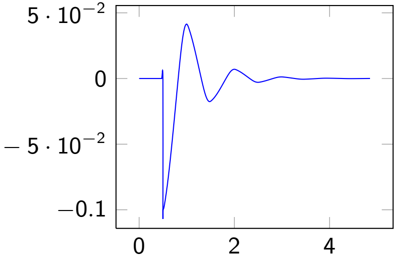
Simulation settings

- Duration of the simulation: 5 seconds.
- Variable step-size solver: `ode23`.
- Zero-crossing detection activated (non adaptive version).

Road profile



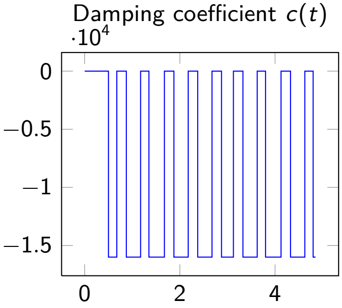
Relative position

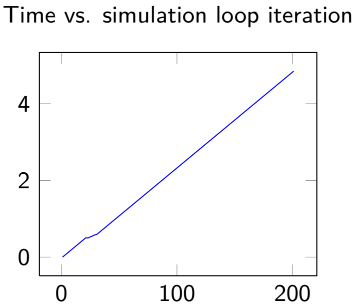


Remarks

- The on-off controller make the suspension stable.

A simple example: simulation results





Remark: numerical precision

The damping force oscillated because of tiny variations of the suspension.
Consequence: finite precision may induce unexpected behaviors.

Remark: time

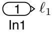
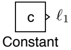
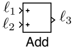
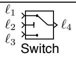
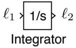
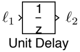
The time evolution is not homogeneous:
Consequence: blocks depending on time may produce more or less output following the time evolution e.g. sine block.

Operational Semantics

Simulink as an equation-based programming language

Input/Output relation

Each block defines the time invariant relation between its input and its output.

Library	Blocks	Representation	Equations
Sources	Input		$\ell_1 = \text{in1}$
	Constant		$\ell_1 = c$
Arithmetic	Add		$\ell_3 = \ell_1 + \ell_2$
Signal routing	Switch		$\ell_4 = \text{if}\,(p_r(\ell_2), \ell_1, \ell_3)$
Continuous-time	Integrator		$\{\ell_2 = x; \dot{x} = \ell_1; x(0) = \text{init}\}$
Discrete-time	Unit Delay		$\{\ell_2 = d; \bar{d} =_S \ell_1; d(0) = \text{init}\}$

$\bar{d} =_S \ell$ stands for at each $t = kS$, $d = \ell$ else it keeps its previous value.

Simulink as an equation-based programming language

Input/Output relation

Each block defines the time invariant relation between its input and its output.

A core language of equations

$$e ::= r \mid \ell \mid x \mid d \mid e_1 \diamond e_2 \mid e_1 \bowtie e_2 \mid \mathrm{if}\big(e_1, e_2, e_3\big) \quad (1)$$

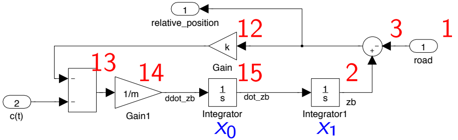
$$eq ::= \ell :=_S e \mid \ell := e \mid \dot{x} := e \mid \bar{d} :=_S e \quad (2)$$

$$p ::= eq \mid eq; p \quad (3)$$

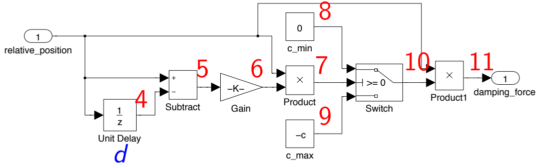
with

- $r \in \mathbb{R}$, constant values;
- $\ell \in \mathcal{V}$, variables associated to a **block output**;
- $x \in \mathcal{V}$, variables associated to **continuous-time states**;
- $d \in \mathcal{V}$, variables associated to **discrete-time states**;
- $\diamond \in \{+, -, \times, \div\}$, arithmetic operations;
- $\bowtie \in \{<, \leq, >, \geq, =, <>\}$, relational operations;
- S is the set of all the sampling times.

Simulink as an equation-based programming language



In red: evaluation order of blocks



For each block in the evaluation order, a simple translation gives:

$$
\begin{aligned}
\ell_1 &= \text{input} \\
\dot{x}_1 &= \ell_{15} \\
\ell_2 &= x_1 \\
\ell_3 &= \ell_2 - \ell_1 \\
\bar{d} &=_S \ell_3 \\
\ell_4 &= d
\end{aligned}
$$

$$
\begin{aligned}
\ell_5 &= \ell_3 - \ell_4 \\
\ell_6 &= 1/40 \times \ell_5 \\
\ell_7 &= \ell_3 \times \ell_6 \\
\ell_8 &= 0 \\
\ell_9 &= -16000 \\
\ell_{10} &= \text{if } \ell_7 \geq 0 \text{ then } \ell_8 \text{ else } \ell_9
\end{aligned}
$$

$$
\begin{aligned}
\ell_{11} &= \ell_3 \times \ell_{10} \\
\ell_{12} &= 20000 \times \ell_3 \\
\ell_{13} &= -\ell_{12} - \ell_{11} \\
\ell_{14} &= 1/250 \times \ell_{13} \\
\dot{x}_0 &= \ell_{14} \\
\ell_{15} &= x_0
\end{aligned}
$$

Simulink as an equation-based programming language

Kinds of equations

A Simulink model is made of four kinds of functions:

- the output function;
- the update function of discrete-time states;
- the update function of continuous-time states;

$$
\begin{aligned}
\ell_1 &= \text{input} \\
\dot{x}_1 &= \ell_{15} \\
\ell_2 &= x_1 \\
\ell_3 &= \ell_2 - \ell_1 \\
\bar{d} &=_S \ell_3 \\
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\ell_{14} &= \frac{1}{250} \times \ell_{13} \\
\dot{x}_0 &= \ell_{14} \\
\ell_{15} &= x_0
\end{aligned}
$$

Remark

These functions allow a **state-space representation** of a model.

Simulink as an equation-based programming language

Kinds of equations

A Simulink model is made of four kinds of functions:

- the output function;
- the update function of discrete-time states;
- the update function of continuous-time states;

$\ell_1 = \text{input}$

$\dot{x}_1 = \ell_{15}$

$\ell_2 = x_1$

$\ell_3 = \ell_2 - \ell_1$

$\bar{d} =_S \ell_3$

$\ell_4 = d$

$\ell_5 = \ell_3 - \ell_4$

$\ell_6 = 1/40 \times \ell_5$

$\ell_7 = \ell_3 \times \ell_6$

$\ell_8 = 0$

$\ell_9 = -16000$

$\ell_{10} = \text{if } \ell_7 \geq 0 \text{ then } \ell_8 \text{ else } \ell_9$

$\ell_{11} = \ell_3 \times \ell_{10}$

$\ell_{12} = 20000 \times \ell_3$

$\ell_{13} = -\ell_{12} - \ell_{11}$

$\ell_{14} = \frac{1}{250} \times \ell_{13}$

$\dot{x}_0 = \ell_{14}$

$\ell_{15} = x_0$

Notation

We denote this function by $\mathbf{g}(\mathbf{t}, \mathbf{x}, \mathbf{d})$.

Simulink as an equation-based programming language

Kinds of equations

A Simulink model is made of four kinds of functions:

- the output function;
- the update function of discrete-time states;
- the update function of continuous-time states;

$\ell_1 = \text{input}$

$\dot{x}_1 = \ell_{15}$

$\ell_2 = x_1$

$\ell_3 = \ell_2 - \ell_1$

$\bar{d} =_S \ell_3$

$\ell_4 = d$

$\ell_5 = \ell_3 - \ell_4$

$\ell_6 = 1/40 \times \ell_5$

$\ell_7 = \ell_3 \times \ell_6$

$\ell_8 = 0$

$\ell_9 = -16000$

$\ell_{10} = \text{if } \ell_7 \geq 0 \text{ then } \ell_8 \text{ else } \ell_9$

$\ell_{11} = \ell_3 \times \ell_{10}$

$\ell_{12} = 20000 \times \ell_3$

$\ell_{13} = -\ell_{12} - \ell_{11}$

$$\ell_{14} = \frac{1}{250} \times \ell_{13}$$

$\dot{x}_0 = \ell_{14}$

$\ell_{15} = x_0$

Notation

We denote this function by $\mathbf{f_d}(\mathbf{t}, \mathbf{x}, \mathbf{d}) = g(t, x, d) \mid_{\bar{d} =_S \ell}$.

Simulink as an equation-based programming language

Kinds of equations

A Simulink model is made of four kinds of functions:

- the output function; (2 versions: major g and **minor** $\tilde{g}$)
- the update function of discrete-time states;
- the update function of continuous-time states;

$$\ell_1 = \text{input}$$
$$\dot{x}_1 = \ell_{15}$$
$$\ell_2 = x_1$$
$$\ell_3 = \ell_2 - \ell_1$$
$$\bar{d} =_S \ell_3$$
$$\ell_4 = d$$

$$\ell_5 = \ell_3 - \ell_4$$
$$\ell_6 = 1/40 \times \ell_5$$
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$$\ell_{11} = \ell_3 \times \ell_{10}$$
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$$\ell_{13} = -\ell_{12} - \ell_{11}$$
$$\ell_{14} = \frac{1}{250} \times \ell_{13}$$
$$\dot{x}_0 = \ell_{14}$$
$$\ell_{15} = x_0$$

Notation

We denote this function by $\mathbf{f_x}(\mathbf{t}, \mathbf{x}, \mathbf{d}) = \tilde{g}(t, x, d) \mid_{\dot{d} =_S \ell}$.

Simulink as an equation-based programming language

Kinds of equations

A Simulink model is made of four kinds of functions:

- the output function;
- the update function of discrete-time states;
- the update function of continuous-time states;

Hidden equations: the 5th kind

Some blocks are associated to equations to detect **zero-crossing events** (only with variable step solvers).

Example

`Switch block`: detect when the sign of the 2nd input changes.

Notation

We denote a zero-crossing equation $f_z(t, x, d)$.

Overview of numerical simulation

Goal: computing the **temporal evolution** of the system.

The steps of the Simulink's simulation engine

Input: $\mathbf{x}_0$, $\mathbf{d}_0$, t_0, h_0;
$n = 0$;
loop until $t_n \geq t_{\text{end}}$
    evaluate $g(t_n, \mathbf{x}_n, \mathbf{d}_n)$;
    update $\mathbf{d}' = f_d(t_n, \mathbf{x}_n, \mathbf{d}_n)$;
    solve $\dot{\mathbf{x}}(t) = f_x(t, \mathbf{x}(t), \mathbf{d}_n)$ over interval $[t_n, t_n + h_n]$ to get $\mathbf{x}(t_n + h_n)$;
    find_zero_crossing ;
    compute h_{n+1} ;
    compute t_{n+1} ;
    $\mathbf{d}_{n+1} = \mathbf{d}'$; $\mathbf{x}_{n+1} = \mathbf{x}(t_n + h_n)$; $n = n + 1$;
end loop

Remarks

- Major steps: evaluation of g produces the simulation results at t_n.
- Minor steps: evaluations of $\tilde{g}$ are used as intermediate computations between t_n and $t_n + h_n$.

Overview of numerical simulation

The **temporal evolution** of the system depends on a set of parameters.

Parameters (a sample)

A Simulink model is described by a set of parameters:

- $t_0 = 0$ **start time** of the simulation;
- $t_{\text{end}} = 10$ **stop time** of the simulation;
- **numerical integration methods** with an **absolute tolerance** ($\texttt{atol} = 10^{-6}$), a **relative tolerance** ($\texttt{rtol} = 10^{-3}$);
- **minimal** ($\texttt{hmin} = 0.2$) and **maximal** ($\texttt{hmax} = 5$) integration step-size;
- **zero-crossing method**: adaptive or non adaptive;
- zero-crossing tolerance ($\texttt{zctol} = 10 \times 128 \times \text{eps}$).

Notation

All these parameters are represented by a record π : name $\rightarrow$ value.

Consequence

There are several semantics of Simulink.

Operational semantics of Simulink

...

loop until $t_n \geq t_{\text{end}}$

  evaluate $g(t_n, \mathbf{x}_n, \mathbf{d}_n)$;

  update $\mathbf{d}' = f_d(t_n, \mathbf{x}_n, \mathbf{d}_n)$;

  solve $\dot{\mathbf{x}}(t) = f_x(t, \mathbf{x}(t), \mathbf{d}_n)$ over interval $[t_n, t_n + h_n]$ to get $\mathbf{x}(t_n + h_n)$;

  find_zero_crossing ;

  ...

end loop

Main rules

$$\frac{\sigma(t) = \pi(t_{\text{end}})}{\mathrm{Eq}, \pi \vdash \sigma \Rightarrow \sigma} \ \textsc{Simulation-End}$$

$$\frac{\sigma(t) < \pi(t_{\text{end}}) \qquad \mathrm{Eq}, \pi \vdash \sigma \xrightarrow{M} \sigma_1 \qquad \mathrm{Eq}, \pi \vdash \sigma_1 \xrightarrow{u} \sigma_2 \qquad \mathrm{Eq}, \pi \vdash \sigma_2 \xrightarrow{s} \sigma'}{\mathrm{Eq}, \pi \vdash \sigma \Rightarrow \sigma'}$$

Notation

$$\mathrm{Eq} = \left\{ g(t, x, d), f_d(t, x, d), f_x(t, x, d), f_z(t, x, d) \right\}$$

Rules for output function

Recall: main rules

$$\frac{\sigma(t) < \pi(t_{\mathrm{end}}) \qquad \mathrm{Eq}, \pi \vdash \sigma \xrightarrow{\mathsf{M}} \sigma_1 \qquad \mathrm{Eq}, \pi \vdash \sigma_1 \xrightarrow{u} \sigma_2 \qquad \mathrm{Eq}, \pi \vdash \sigma_2 \xrightarrow{s} \sigma'}{\mathrm{Eq}, \pi \vdash \sigma \Rightarrow \sigma'}$$

Very simple rules of a classical operational semantics.

Step

- Given values of continuous states x , values of discrete states d and input values.
- Evaluate each equations given by $g(t, x, d)$ in the evaluation order

Sample of evaluation rules

$$\frac{\langle e_1, \sigma \rangle \xrightarrow{o} r_1 \qquad \langle e_2, \sigma \rangle \xrightarrow{o} r_2}{\langle e_1 \diamond e_2, \sigma \rangle \xrightarrow{o} r_1 \diamond r_2} \text{ ARITH} \qquad \frac{\langle e_1, \sigma \rangle \xrightarrow{o} \mathsf{true} \qquad \langle e_2, \sigma \rangle \xrightarrow{o} r_2}{\langle \mathsf{if}\ (e_1, e_2, e_3), \sigma \rangle \xrightarrow{o} r_2} \text{ THEN}$$

Rules for update discrete states

Recall: main rules

$$\frac{\sigma(t) < \pi(t_{\text{end}}) \qquad \text{Eq}, \pi \vdash \sigma \xrightarrow{M} \sigma_1 \qquad \text{Eq}, \pi \vdash \sigma_1 \xrightarrow{\mathbf{u}} \sigma_2 \qquad \text{Eq}, \pi \vdash \sigma_2 \xrightarrow{s} \sigma'}{\text{Eq}, \pi \vdash \sigma \Rightarrow \sigma'}$$

Main idea:

- if the current time **t** is a sampling time then update the state.
- otherwise keep the previous value.

Simple rules

$$\frac{\text{Eq}(\mathbf{d}) = \overline{\mathbf{d}} :=_S \ell \qquad \sigma(t) \in S}{\text{Eq}, \pi \vdash \sigma \xrightarrow{u} \sigma[\mathbf{d} \mapsto \sigma(\ell)]} \qquad \frac{\text{Eq}(\mathbf{d}) = \overline{\mathbf{d}} :=_S \ell \qquad \sigma(t) \notin S}{\text{Eq}, \pi \vdash \sigma \xrightarrow{u} \sigma}$$

Remark

The numerical integration is in charge not to miss a sampling time.

Rules for update continuous states

Recall: Main rules

$$\frac{\sigma(t) < \pi(t_{\text{end}}) \qquad \text{Eq}, \pi \vdash \sigma \xrightarrow{M} \sigma_1 \qquad \text{Eq}, \pi \vdash \sigma_1 \xrightarrow{u} \sigma_2 \qquad \text{Eq}, \pi \vdash \sigma_2 \xrightarrow{s} \sigma'}{\text{Eq}, \pi \vdash \sigma \Rightarrow \sigma'}$$

The main features of the Simulink solver is encoded in this rules:

- Performing numerical integration.
- Handling zero-crossing.
- Adapting integration step-size.

Sub rules

$$\frac{\text{Eq}, \pi, \sigma \vdash \sigma \xrightarrow{i} \sigma_{so} \qquad \text{Eq}, \pi, \sigma \vdash \sigma_{so} \xrightarrow{zc} \sigma_{zc} \qquad \text{Eq}, \pi \vdash \sigma_{zc} \xrightarrow{h} \sigma'}{\text{Eq}, \pi \vdash \sigma \xrightarrow{s} \sigma'}$$

Remarks

Hypotheses to enforce the existence of the ODEs solution.

- No discrete states are updated during numerical integration.
- Numerical integration assumes no zero-crossing events appear.

Numerical integration methods – 1

There exists a huge amount of methods which may categorize in:

- explicit or implicit method;
- fixed or variable step-size;
- single-step or multi-step method;
- fixed or variable order.

Simulink's integration methods considered

Explicit single-step fixed step-size

- ode1 (Euler)
- ode2 (Heun)
- ode3 (Bogacki-Shampine)
- ode4 (Dormand-Prince)
- ode8 (Dormand-Prince)

Explicit single-step variable step-size

- ode23 (Bogacki-Shampine)
- ode45 (Dormand-Prince)

Remark

All these methods are members of the Runge-Kutta family which has a unified representation with **Butcher Table**.

Numerical integration methods – 2

Example

Simulink solver `ode23`

Mathematical description

$$k_1 = f(t_n, x_n)$$

$$k_2 = f(t_n + \tfrac{1}{2} h_n, x_n + \tfrac{1}{2} h k_1)$$

$$k_3 = f(t_n + \tfrac{3}{4} h_n, x_n + \tfrac{3}{4} h k_2)$$

$$x_{n+1} = x_n + h\left(\tfrac{2}{9} k_1 + \tfrac{1}{3} k_2 + \tfrac{4}{9} k_3\right)$$

$$k_4 = f(t_n + 1\, h_n, x_{n+1})$$

$$y_{n+1} = x_n + h\left(\tfrac{7}{24} k_1 + \tfrac{1}{4} k_2 + \tfrac{1}{3} k_3 + \tfrac{1}{8} k_4\right)$$

Butcher Table

$$\begin{array}{c|cccc}
0 & & & & \\
\frac{1}{2} & \frac{1}{2} & & & \\
\frac{3}{4} & 0 & \frac{3}{4} & & \\
1 & \frac{2}{9} & \frac{1}{3} & \frac{4}{9} & \\
\hline
 & \frac{2}{9} & \frac{1}{3} & \frac{4}{9} & 0 \\
 & \frac{7}{24} & \frac{1}{4} & \frac{1}{3} & \frac{1}{8}
\end{array}$$

Remark

The parameters set π embeds the Butcher Table of the considered integration method $\Rightarrow$ **handling of the several semantics of Simulink**.

Numerical integration methods – 2

Example

Simulink solver `ode23`

Mathematical description

$$k_1 = f(t_n, x_n)$$
$$k_2 = f(t_n + \tfrac{1}{2} h_n, x_n + \tfrac{1}{2} h k_1)$$
$$k_3 = f(t_n + \tfrac{3}{4} h_n, x_n + \tfrac{3}{4} h k_2)$$
$$x_{n+1} = x_n + h\left(\tfrac{2}{9} k_1 + \tfrac{1}{3} k_2 + \tfrac{4}{9} k_3\right)$$
$$k_4 = f(t_n + 1 h_n, x_{n+1})$$
$$y_{n+1} = x_n + h\left(\tfrac{7}{24} k_1 + \tfrac{1}{4} k_2 + \tfrac{1}{3} k_3 + \tfrac{1}{8} k_4\right)$$

Simulink translation

$$k_1 = f_x(t_n, x_n, d_n);$$
$$x = x_n + \tfrac{1}{2} k_1;\ t = t_n + \tfrac{1}{2} h_n;$$
$$k_2 = f_x(t, x, d_n);$$
$$x = x_n + \tfrac{3}{4} k_1;$$
$$\ldots$$
$$x_{n+1} = x_n + h\left(\tfrac{2}{9} k_1 + \tfrac{1}{3} k_2 + \tfrac{4}{9} k_3\right);$$
$$\ldots$$

Consequence

Our semantic rules act as an evaluation of a local expansion of the sequence of equations.

Numerical integration methods – 2

Evaluation rules for variable step-size solver

$$\frac{\begin{array}{c}\langle \mathtt{sc}_1(\sigma_M(\mathbf{x}), \mathsf{Eq}(\mathbf{x}), \pi)\ ,\ \sigma\rangle \xrightarrow{o} \sigma_1 \\ \langle \mathtt{sc}_2(\sigma_M(\mathbf{x}), \mathsf{Eq}(\mathbf{x}), \pi)\ ,\ \sigma_1\rangle \xrightarrow{o} \sigma_2 \qquad \mathtt{check\_err}(\sigma, \sigma_1, \sigma_2, \pi) = 1\end{array}}{\mathsf{Eq}, \pi, \sigma_M \vdash \sigma \xrightarrow{i} \sigma_M[\mathbf{x} \mapsto \sigma_1(\mathbf{x}),\ h \mapsto \sigma_1(h)]} \text{ INTEGR-SUCCESS}$$

$$\frac{\begin{array}{c}\langle \mathtt{sc}_1(\sigma_M(\mathbf{x}), \mathsf{Eq}(\mathbf{x}), \pi)\ ,\ \sigma\rangle \xrightarrow{o} \sigma_1 \\ \langle \mathtt{sc}_2(\sigma_M(\mathbf{x}), \mathsf{Eq}(\mathbf{x}), \pi)\ ,\ \sigma_1\rangle \xrightarrow{o} \sigma_2 \qquad \mathtt{check\_err}(\sigma, \sigma_1, \sigma_2, \pi) = 0 \\ \mathsf{Eq}, \pi, \sigma_M \vdash \sigma_M[h \mapsto \max(\pi(h_{\min}), \frac{\sigma(h)}{2})] \xrightarrow{i} \sigma'\end{array}}{\mathsf{Eq}, \pi, \sigma_M \vdash \sigma \xrightarrow{i} \sigma'} \text{ INTEGR-FAIL}$$

Note: sc_i represents the local expansion of equations.

Solving ODEs leads to a time discretization

Equations of the form $\dot{x}_i = \ell_j$ are substituted by the **sequence of the equations** needed to realize the numerical integration.

Numerical integration methods – 3

Validation of the integration step

For adaptive step-size method: for all continuous state variables

$$\text{integration error} = \frac{h_n \parallel x_{n+1} - y_{n+1} \parallel_\infty}{\max\left(\max\left(|x_{n+1}|, |x_n|\right), \frac{atol}{rtol}\right)} \overset{\text{valid if}}{\leq} \text{rtol} \ .$$

Strategy:

- **Success:** go the the zero-crossing detection step.
- **Failure:** reduce the step-size h_n in general only a division by 2. and restart the integration step with the new step-size.

Remark

The reduction of the step-size is done until the $h_{\min}$ is reached. In that case a simulation error may happen.

Zero-crossing event detection (non adaptive version)

Main steps

- **Detection** of zero-crossing event
  Is one of the zero-crossing changed its sign between $[t_n, t_n + h_n]$?
- **Localization**: if detection is true
  Bracket the most recent zero-crossing time using bisection method.
- **Pass through** the zero-crossing event in two steps:
  - Set the next major output to the left bound of the bracket time.
  - Reset the solver with the state estimate at the right bound of bracket time.

Ingredients for the localization – 1

Linear approximation of the dynamic.

$$\texttt{computeTz}(\sigma_L, \sigma_R) = \min_L \left\{ \sigma_L(t) - x_L \cdot \frac{\sigma_R(t) - \sigma_L(t)}{x_R - x_L} : \forall x \right\}$$

with $\min_L(t) = \min\{t_i \mid t_i \geq \sigma_L(t)\}$.
Remark: σ_L and σ_R represents the environments at the time enclosing the event.

Zero-crossing event detection (non adaptive version)

Main steps

- **Detection** of zero-crossing event
  Is one of the zero-crossing changed its sign between $[t_n, t_n + h_n]$?
- **Localization**: if detection is true
  Bracket the most recent zero-crossing time using bisection method.
- **Pass through** the zero-crossing event in two steps:
  - Set the next major output to the left bound of the bracket time.
  - Reset the solver with the state estimate at the right bound of bracket time.

Ingredients for the localization – 2

Continuous extension (method dependent) to easily estimate state.
For example, `ode23`

$$\begin{aligned}\texttt{interpolX}(\sigma_{t_n}, \sigma_{t_n+h_n}, t) = {} & (2\tau^3 - 3\tau^2 + 1)\sigma_{t_n}(x) + (\tau^3 - 2\tau^2 + \tau)(t_2 - t_1)p_1 \\ & + (-2\tau^3 + 3\tau^2)\sigma_{t_n+h_n}(x) + (\tau^3 - \tau^2)(t_2 - t_1)p_2\end{aligned}$$

with $\tau = \frac{t - t_n}{h_n}$

Update integration step-size

Goals of this step

Performance issue:

- increase the step-size to reduce the number of simulation loop.
- or decrease the step-size to increase the accuracy.

Scheduling issue: must not missed a discrete sampling time.

Main rule

$$\frac{h' = \texttt{changeH}(\sigma(h), \pi) \qquad t' = \min\{\tau \in \mathsf{Eq}(\texttt{sampling}) : \tau > \sigma(t)\}}{\mathsf{Eq}, \pi, \sigma_M \vdash \sigma \xrightarrow{h} \sigma[h \mapsto \min(h', t' - \sigma(t))]}$$

Note: `changeH` increase or decrease h in function of the integration error.

Remark

If a zero-crossing occurred the step-size is left unchanged in regards to the performance issue.

Conclusion

Conclusion and future work

Conclusion

Despite Simulink is a closed source software, **the main algorithms in the solver are known**.

Hope

The definition of a formal semantics will help increasing the confidence in the Simulink design flow with new verification methods.

Future work

- **Pursuing the formalization** of the semantics: reset state, adaptive zero-crossing, integration method like `ode113`, etc.
- **Development of a homemade simulator** of Simulink models to validate experimentally our semantics.
- **Development of a set-based simulator** in order to address the reachability problem of Simulink models.