Towards an Operational Semantics of the Simulation Engine of Simulink

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Motivation
Simulink and its industrial uses

Context

**Simulink** is a *de facto* standard in the industry to design embedded safety critical applications.

e.g. it is used to design almost all the **drive-by-wire** systems.

Components of a drive-by-wire systems

- A **physical mechanism** we want to control, e.g. a suspension.
- A **software** which implement the controller.

⇒ we have to deal with **hybrid systems**.

Current validation method

The main method to validate the design of embedded systems is based on **simulation activity**.
Mathematical models are an approximated descriptions of physical systems.

The computer descriptions are studied with numerical tools.
**MIL: Model in the loop**

Definition of a mathematical model of the plant and the controller.

Aim of the testing activity

- Does the controller fulfil the specification?
- The set of tests will be used as “an oracle” in the next steps.
Classical design and validation methodology

**SIL: Software in the loop**
Implementation of the controller in a target language.

Aim of the testing activity
- Do the hand-written or generated code still fulfil the specification?
**PIL: Processor in the loop**

Compilation of the controller and execute it on a virtual processor.

Test vectors → Test process → Virtual targeted CPU

Controller: object code

Sensor model

Plant Model

Actuator model

Expected output → Simulation platform

Aim of the testing activity

- Does the low-level implementation still fulfil the specification?
Simulink: a graphical language

It allows to describe, as block-diagrams, a mixing of:

- **Ordinary differential equations** (ODE).
- Finite difference equations: **single-rate** or **multi-rate** period.
- Conditional executed equations: **enabled** or **triggered**.

Simulink: several parametrized semantics

- the definition of a **numerical solver** used to solve ODEs;
- the detection of events: **zero-crossing**.

So, it is a **numerical approximation** of the **mathematical behavior**.

What we should keep in mind!

The **designer** only refers to the **Simulink’s semantics** (i.e. graphical output) to **validate** the development steps!
Formal verification of Simulink

**Fact:** simulations will never be complete and cannot bring strong guarantee on the designed software.

**Main question:** How can we help designer to be more confident in its designed systems with formal methods?

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**Formal verification of Simulink: Our Goal**

- To compute invariant properties on the software **taking into account** a model of the plant.
- **Our approach:** abstract interpretation-based static analysis.

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**Formal verification of Simulink: Constraint**

- Formal verification methods should be adaptable for the design process in order to be more easily adopted by the end-user.

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**Formal verification of Simulink: Challenge**

- Understand the **simulation engine** of Simulink.
Simulink Language
A simple example: mathematical models

A semi-active suspension of a quarter-car model

Mathematical model of the mechanical system
\[ \ddot{z}_b = -\frac{1}{m} \left( k(z_b - z_r) + c(t) \right). \]

Mathematical model of the controller
\[ c(t) = \begin{cases} -c_{\text{max}}(z_b - z_r) & \text{if } (z_b - z_r)(\dot{z}_b - \dot{z}_r) < 0 \\ c_{\text{min}} & \text{if } (z_b - z_r)(\dot{z}_b - \dot{z}_r) \geq 0 \end{cases}. \]

- \( m = 250 \text{ kg} \)
- \( k = 20000 \text{ N/m} \)
- \( c_{\text{max}} = 16000 \text{ N/m/s} \)
- \( c_{\text{min}} = 0 \)
A simple example: Simulink implementation

Mathematical model of the mechanical system

\[ \ddot{z}_b = -\frac{1}{m} \left( k(z_b - z_r) + c(t) \right). \]

Integrator block

Associated to a first order dynamic system:

\[
\begin{align*}
\dot{x}(t) &= \text{input}(t) \\
\text{output}(t) &= x(t) \quad \text{with} \quad x(0) = x_0
\end{align*}
\]
Mathematical model of the controller

\[ c(t) = \begin{cases} 
-c_{\text{max}}(z_b - z_r) & \text{if } (z_b - z_r)(\dot{z}_b - \dot{z}_r) < 0 \\
c_{\text{min}} & \text{if } (z_b - z_r)(\dot{z}_b - \dot{z}_r) \geq 0 
\end{cases}. \]

**Implementation:** \((\dot{z}_b - \dot{z}_r)\) is given by differentiating the sensor output.

Closed-loop system
- Connect the output of the plant to the input of the controller.
- Connect the output of the controller to the input of the plant.

Discrete differentiation at rate 1/40 sec.
A simple example: simulation results

Simulation settings
- Duration of the simulation: 5 seconds.
- Zero-crossing detection activated (non adaptive version).

Road profile

Relative position

Remarks
- The on-off controller make the suspension stable.
A simple example: simulation results

Remark: numerical precision

The damping force oscillated because of tiny variations of the suspension. **Consequence:** finite precision may induce unexpected behaviors.

Remark: time

The time evolution is not homogeneous: **Consequence:** blocks depending on time may produce more or less output following the time evolution e.g. sine block.
Simulink as an equation-based programming language

**Input/Output relation**

Each block defines the time invariant relation between its input and its output.

<table>
<thead>
<tr>
<th>Library</th>
<th>Blocks</th>
<th>Representation</th>
<th>Equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sources</td>
<td>Input</td>
<td><img src="image" alt="Input Block" /></td>
<td>$\ell_1 = \text{in1}$</td>
</tr>
<tr>
<td></td>
<td>Constant</td>
<td><img src="image" alt="Constant Block" /></td>
<td>$\ell_1 = c$</td>
</tr>
<tr>
<td>Arithmetic</td>
<td>Add</td>
<td><img src="image" alt="Add Block" /></td>
<td>$\ell_3 = \ell_1 + \ell_2$</td>
</tr>
<tr>
<td>Signal routing</td>
<td>Switch</td>
<td><img src="image" alt="Switch Block" /></td>
<td>$\ell_4 = \text{if} \left( p_r(\ell_2), \ell_1, \ell_3 \right)$</td>
</tr>
<tr>
<td>Continuous-time</td>
<td>Integrator</td>
<td><img src="image" alt="Integrator Block" /></td>
<td>${ \ell_2 = x; \dot{x} = \ell_1; x(0) = \text{init} }$</td>
</tr>
<tr>
<td>Discrete-time</td>
<td>Unit Delay</td>
<td><img src="image" alt="Unit Delay Block" /></td>
<td>${ \ell_2 = d; \bar{d} = \ell_1; d(0) = \text{init} }$</td>
</tr>
</tbody>
</table>

$\bar{d} = \_S \ell$ stands for at each $t = kS$, $d = \ell$ else it keeps its previous value.
Simulink as an equation-based programming language

**Input/Output relation**

Each block defines the time invariant relation between its input and its output.

**A core language of equations**

\[
e ::= r | \ell | x | d | e_1 \diamond e_2 | e_1 \bowtie e_2 | \text{if}(e_1, e_2, e_3) \tag{1}
\]
\[
eq ::= \ell :=_S e | \ell := e | \dot{x} := e | \bar{d} :=_S e \tag{2}
\]
\[
p ::= eq | eq; p \tag{3}
\]

with

- \( r \in \mathbb{R} \), constant values;
- \( \ell \in \mathcal{V} \), variables associated to a **block output**;
- \( x \in \mathcal{V} \), variables associated to **continuous-time states**;
- \( d \in \mathcal{V} \), variables associated to **discrete-time states**;
- \( \diamond \in \{+,-,\times,\div\} \), arithmetic operations;
- \( \bowtie \in \{<,\leq,>,\geq,=,<>\} \), relational operations;
- \( S \) is the set of all the sampling times.
Simulink as an equation-based programming language

In red: evaluation order of blocks

For each block in the evaluation order, a simple translation gives:

\[ \ell_1 = \text{input} \]
\[ \dot{x}_1 = \ell_{15} \]
\[ \ell_2 = x_1 \]
\[ \ell_3 = \ell_2 - \ell_1 \]
\[ \ddot{d} = s \ell_3 \]
\[ \ell_4 = d \]
\[ \ell_5 = \ell_3 - \ell_4 \]
\[ \ell_6 = \frac{1}{40} \times \ell_5 \]
\[ \ell_7 = \ell_3 \times \ell_6 \]
\[ \ell_8 = 0 \]
\[ \ell_9 = -16000 \]
\[ \ell_{10} = \text{if } \ell_7 \geq 0 \text{ then } \ell_8 \text{ else } \ell_9 \]
\[ \ell_{11} = \ell_3 \times \ell_{10} \]
\[ \ell_{12} = 20000 \times \ell_3 \]
\[ \ell_{13} = -\ell_{12} - \ell_{11} \]
\[ \ell_{14} = \frac{1}{250} \times \ell_{13} \]
\[ \dot{x}_0 = \ell_{14} \]
\[ \ell_{15} = x_0 \]
Kinds of equations

A Simulink model is made of four kinds of functions:

- the output function;
- the update function of discrete-time states;
- the update function of continuous-time states;

\[
\begin{align*}
\ell_1 &= \text{input} \\
\dot{x}_1 &= \ell_{15} \\
\ell_2 &= x_1 \\
\ell_3 &= \ell_2 - \ell_1 \\
\bar{d} &= \ell_3 \\
\ell_4 &= d \\
\ell_5 &= \ell_3 - \ell_4 \\
\ell_6 &= 1/40 \times \ell_5 \\
\ell_7 &= \ell_3 \times \ell_6 \\
\ell_8 &= 0 \\
\ell_9 &= -16000 \\
\ell_{10} &= \text{if } \ell_7 \geq 0 \text{ then } \ell_8 \text{ else } \ell_9 \\
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\ell_{13} &= -\ell_{12} - \ell_{11} \\
\ell_{14} &= \frac{1}{250} \times \ell_{13} \\
\dot{x}_0 &= \ell_{14} \\
\ell_{15} &= x_0
\end{align*}
\]

Remark

These functions allow a state-space representation of a model.
Kinds of equations

A Simulink model is made of four kinds of functions:
- the output function;
- the update function of discrete-time states;
- the update function of continuous-time states;

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\begin{align*}
\ell_1 &= \text{input} \\
\dot{x}_1 &= \ell_{15} \\
\ell_2 &= x_1 \\
\ell_3 &= \ell_2 - \ell_1 \\
\bar{d} &= \ell_3 \\
\ell_4 &= d \\
\ell_5 &= \ell_3 - \ell_4 \\
\ell_6 &= \frac{1}{40} \times \ell_5 \\
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\ell_8 &= 0 \\
\ell_9 &= -16000 \\
\ell_{10} &= \text{if } \ell_7 \geq 0 \text{ then } \ell_8 \text{ else } \ell_9 \\
\ell_{11} &= \ell_3 \times \ell_{10} \\
\ell_{12} &= 20000 \times \ell_3 \\
\ell_{13} &= -\ell_{12} - \ell_{11} \\
\ell_{14} &= \frac{1}{250} \times \ell_{13} \\
\dot{x}_0 &= \ell_{14} \\
\ell_{15} &= x_0
\end{align*}
\]

Notation

We denote this function by \( g(t, x, d) \).
Simulink as an equation-based programming language

Kinds of equations

A Simulink model is made of four kinds of functions:

- the output function;
- the update function of discrete-time states;
- the update function of continuous-time states;

\[
\begin{align*}
\ell_1 &= \text{input} & \ell_5 &= \ell_3 - \ell_4 & \ell_{11} &= \ell_3 \times \ell_{10} \\
\dot{x}_1 &= \ell_{15} & \ell_6 &= 1/40 \times \ell_5 & \ell_{12} &= 20000 \times \ell_3 \\
\ell_2 &= x_1 & \ell_7 &= \ell_3 \times \ell_6 & \ell_{13} &= -\ell_{12} - \ell_{11} \\
\ell_3 &= \ell_2 - \ell_1 & \ell_8 &= 0 & \ell_{14} &= \frac{1}{250} \times \ell_{13} \\
\bar{d} &= \ell_3 & \ell_9 &= -16000 & \dot{x}_0 &= \ell_{14} \\
\ell_4 &= d & \ell_{10} &= \text{if } \ell_7 \geq 0 \text{ then } \ell_8 \text{ else } \ell_9 & \ell_{15} &= x_0
\end{align*}
\]

Notation

We denote this function by \( f_d(t, x, d) = g(t, x, d) \mid \bar{d} = s \ell \).
Simulink as an equation-based programming language

Kinds of equations

A Simulink model is made of four kinds of functions:

- the output function; (2 versions: major $g$ and minor $\tilde{g}$)
- the update function of discrete-time states;
- the update function of continuous-time states;

\[
\begin{align*}
\ell_1 &= \text{input} & \ell_5 &= \ell_3 - \ell_4 & \ell_{11} &= \ell_3 \times \ell_{10} \\
\dot{x}_1 &= \ell_{15} & \ell_6 &= 1/40 \times \ell_5 & \ell_{12} &= 20000 \times \ell_3 \\
\ell_2 &= x_1 & \ell_7 &= \ell_3 \times \ell_6 & \ell_{13} &= -\ell_{12} - \ell_{11} \\
\ell_3 &= \ell_2 - \ell_1 & \ell_8 &= 0 & \ell_{14} &= \frac{1}{250} \times \ell_{13} \\
\tilde{d} &= s_5 \ell_3 & \ell_9 &= -16000 & \dot{x}_0 &= \ell_{14} \\
\ell_4 &= d & \ell_{10} &= \text{if } \ell_7 \geq 0 \text{ then } \ell_8 \text{ else } \ell_9 & \ell_{15} &= x_0 \\
\ell_{14} &= \frac{1}{250} \times \ell_{13} & \dot{x}_0 &= \ell_{14} \\
\ell_{15} &= x_0 &
\end{align*}
\]

Notation

We denote this function by $f_x(t, x, d) = \tilde{g}(t, x, d) \big|_{d=s_5\ell}$. 
Kinds of equations
A Simulink model is made of four kinds of functions:
- the output function;
- the update function of discrete-time states;
- the update function of continuous-time states;

Hidden equations: the 5th kind
Some blocks are associated to equations to detect zero-crossing events (only with variable step solvers).

Example
Switch block: detect when the sign of the 2nd input changes.

Notation
We denote a zero-crossing equation $f_z(t, x, d)$. 
Overview of numerical simulation

**Goal:** computing the *temporal evolution* of the system.

**The steps of the Simulink’s simulation engine**

Input: $x_0$, $d_0$, $t_0$, $h_0$; 
$n = 0$; 
loop until $t_n \geq t_{\text{end}}$

- evaluate $g(t_n, x_n, d_n)$;
- update $d' = f_d(t_n, x_n, d_n)$;
- solve $\dot{x}(t) = f_x(t, x(t), d_n)$ over interval $[t_n, t_n + h_n]$ to get $x(t_n + h_n)$;
- find_zero_crossing;
- compute $h_{n+1}$;
- compute $t_{n+1}$;
- $d_{n+1} = d'$; $x_{n+1} = x(t_n + h_n)$; $n = n + 1$;

end loop

**Remarks**

- **Major steps:** evaluation of $g$ produces the simulation results at $t_n$.
- **Minor steps:** evaluations of $\tilde{g}$ are used as intermediate computations between $t_n$ and $t_n + h_n$. 
Overview of numerical simulation

The **temporal evolution** of the system depends on a set of parameters.

**Parameters (a sample)**

A Simulink model is described by a set of parameters:

- $t_0 = 0$ **start time** of the simulation;
- $t_{\text{end}} = 10$ **stop time** of the simulation;
- **numerical integration methods** with an **absolute tolerance** ($\text{atol} = 10^{-6}$), a **relative tolerance** ($\text{rtol} = 10^{-3}$);
- **minimal** ($h_{\text{min}} = 0.2$) and **maximal** ($h_{\text{max}} = 5$) integration step-size;
- **zero-crossing method**: adaptive or non adaptive;
- zero-crossing tolerance ($zctol = 10 \times 128 \times \epsilon$).

**Notation**

All these parameters are represented by a record $\pi : \text{name} \rightarrow \text{value}$.

**Consequence**

There are several semantics of Simulink.
... loop until $t_n \geq t_{\text{end}}$
    evaluate $g(t_n, x_n, d_n)$;
    update $d' = f_d(t_n, x_n, d_n)$;
    solve $\dot{x}(t) = f_x(t, x(t), d_n)$ over interval $[t_n, t_n + h_n]$ to get $x(t_n + h_n)$;
    find_zero_crossing;
    ... end loop

Main rules

\[
\begin{align*}
\sigma(t) = \pi(t_{\text{end}}) & \quad \text{SIMULATION-END} \\
\sigma(t) < \pi(t_{\text{end}}) & \quad \text{Eq, } \pi \vdash \sigma \Rightarrow \sigma \\
\sigma(t) < \pi(t_{\text{end}}) & \quad \text{Eq, } \pi \vdash \sigma \xrightarrow{M} \sigma_1 \\
\sigma_1 \xrightarrow{u} \sigma_2 & \quad \text{Eq, } \pi \vdash \sigma_1 \xrightarrow{u} \sigma_2 \\
\sigma_2 \xrightarrow{s} \sigma' & \quad \text{Eq, } \pi \vdash \sigma_2 \xrightarrow{s} \sigma' \\
\end{align*}
\]

Notation

\[
\text{Eq} = \{ g(t, x, d), f_d(t, x, d), f_x(t, x, d), f_z(t, x, d) \} 
\]
Rules for output function

Recall: main rules

\[ \sigma(t) < \pi(t_{\text{end}}) \]

\[
\begin{align*}
\text{Eq, } \pi \vdash \sigma & \xrightarrow{\text{M}} \sigma_1 \\
\text{Eq, } \pi \vdash \sigma_1 & \xrightarrow{u} \sigma_2 \\
\text{Eq, } \pi \vdash \sigma_2 & \xrightarrow{s} \sigma' \\
\text{Eq, } \pi \vdash \sigma & \Rightarrow \sigma'
\end{align*}
\]

Very simple rules of a classical operational semantics.

Step

- Given values of continuous states \( x \), values of discrete states \( d \) and input values.
- Evaluate each equations given by \( g(t, x, d) \) in the evaluation order

Sample of evaluation rules

\[
\begin{align*}
\langle e_1, \sigma \rangle & \xrightarrow{o} r_1 \\
\langle e_2, \sigma \rangle & \xrightarrow{o} r_2 \\
\langle e_1 \diamond e_2, \sigma \rangle & \xrightarrow{o} r_1 \diamond r_2 \\
\langle e_1, \sigma \rangle & \xrightarrow{o} \text{true} \\
\langle e_2, \sigma \rangle & \xrightarrow{o} r_2 \\
\langle \text{if } (e_1, e_2, e_3), \sigma \rangle & \xrightarrow{o} r_2
\end{align*}
\]

\text{ARITH} \quad \text{THEN}
Rules for update discrete states

Recall: main rules

\[ \sigma(t) < \pi(t_{\text{end}}) \quad \text{Eq, } \pi \vdash \sigma \xrightarrow{M} \sigma_1 \quad \text{Eq, } \pi \vdash \sigma_1 \xrightarrow{u} \sigma_2 \quad \text{Eq, } \pi \vdash \sigma_2 \xrightarrow{s} \sigma' \]

\[ \text{Eq, } \pi \vdash \sigma \Rightarrow \sigma' \]

Main idea:
- if the current time \( t \) is a sampling time then update the state.
- otherwise keep the previous value.

Simple rules

\[ \text{Eq}(d) = \overline{d} := S \ell \quad \sigma(t) \in S \quad \text{Eq, } \pi \vdash \sigma \xrightarrow{u} \sigma[d \mapsto \sigma(\ell)] \]

\[ \text{Eq}(d) = \overline{d} := S \ell \quad \sigma(t) \notin S \quad \text{Eq, } \pi \vdash \sigma \xrightarrow{u} \sigma \]

Remark
The numerical integration is in charge not to miss a sampling time.
Rules for update continuous states

Recall: Main rules

\[ \sigma(t) < \pi(t_{\text{end}}) \quad \text{Eq, } \pi \vdash \sigma \xrightarrow{M} \sigma_1 \quad \text{Eq, } \pi \vdash \sigma_1 \xrightarrow{u} \sigma_2 \quad \text{Eq, } \pi \vdash \sigma_2 \xrightarrow{s} \sigma' \]

\[ \text{Eq, } \pi \vdash \sigma \Rightarrow \sigma' \]

The main features of the Simulink solver is encoded in this rules:

- Performing numerical integration.
- Handling zero-crossing.
- Adapting integration step-size.

Sub rules

\[ \text{Eq, } \pi, \sigma \vdash \sigma \xrightarrow{i} \sigma_{so} \quad \text{Eq, } \pi, \sigma \vdash \sigma_{so} \xrightarrow{zc} \sigma_{zc} \quad \text{Eq, } \pi \vdash \sigma_{zc} \xrightarrow{h} \sigma' \]

\[ \text{Eq, } \pi \vdash \sigma \xrightarrow{s} \sigma' \]

Remarks

**Hypotheses to enforce the existence** of the ODEs solution.

- No discrete states are updated during numerical integration.
- Numerical integration assumes no zero-crossing events appear.
There exists a huge amount of methods which may categorize in:

- explicit or implicit method;
- fixed or variable step-size;
- single-step or multi-step method;
- fixed or variable order.

### Simulink’s integration methods considered

<table>
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<tr>
<th>Explicit single-step fixed step-size</th>
<th>Explicit single-step variable step-size</th>
</tr>
</thead>
<tbody>
<tr>
<td>ode1 (Euler)</td>
<td>ode23 (Bogacki-Shampine)</td>
</tr>
<tr>
<td>ode2 (Heun)</td>
<td>ode45 (Dormand-Prince)</td>
</tr>
<tr>
<td>ode3 (Bogacki-Shampine)</td>
<td></td>
</tr>
<tr>
<td>ode4 (Dormand-Prince)</td>
<td></td>
</tr>
<tr>
<td>ode8 (Dormand-Prince)</td>
<td></td>
</tr>
</tbody>
</table>

### Remark

All these methods are members of the Runge-Kutta family which has a unified representation with **Butcher Table**.
Example

Simulink solver ode23

Mathematical description

\[ k_1 = f(t_n, x_n) \]
\[ k_2 = f(t_n + \frac{1}{2} h_n, x_n + \frac{1}{2} hk_1) \]
\[ k_3 = f(t_n + \frac{3}{4} h_n, x_n + \frac{3}{4} hk_2) \]
\[ x_{n+1} = x_n + h \left( \frac{2}{9} k_1 + \frac{1}{3} k_2 + \frac{4}{9} k_3 \right) \]
\[ k_4 = f(t_n + h_n, x_{n+1}) \]
\[ y_{n+1} = x_n + h \left( \frac{7}{24} k_1 + \frac{1}{4} k_2 + \frac{1}{3} k_3 + \frac{1}{8} k_4 \right) \]

Remark

The parameters set \( \pi \) embeds the Butcher Table of the considered integration method \( \Rightarrow \) handling of the several semantics of Simulink.
Numerical integration methods – 2

Example
Simulink solver ode23

Mathematical description

\[ k_1 = f(t_n, x_n) \]
\[ k_2 = f(t_n + \frac{1}{2} h_n, x_n + \frac{1}{2} hk_1) \]
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\[ x_{n+1} = x_n + h \left( \frac{2}{9} k_1 + \frac{1}{3} k_2 + \frac{4}{9} k_3 \right) \]
\[ k_4 = f(t_n + h_n, x_{n+1}) \]
\[ y_{n+1} = x_n + h \left( \frac{7}{24} k_1 + \frac{1}{4} k_2 + \frac{1}{3} k_3 + \frac{1}{8} k_4 \right) \]

Simulink translation

\[ k_1 = f_x(t_n, x_n, d_n); \]
\[ x = x_n + \frac{1}{2} k_1; t = t_n + \frac{1}{2} h_n; \]
\[ k_2 = f_x(t, x, d_n); \]
\[ x = x_n + \frac{3}{4} k_1; \]
\[ \ldots \]
\[ x_{n+1} = x_n + h \left( \frac{2}{9} k_1 + \frac{1}{3} k_2 + \frac{4}{9} k_3 \right); \]
\[ \ldots \]

Consequence

Our semantic rules act as an evaluation of a local expansion of the sequence of equations.
Evaluation rules for variable step-size solver

\[
\begin{align*}
\langle sc_1(\sigma_M(x), \text{Eq}(x), \pi), \sigma \rangle & \xrightarrow{o} \sigma_1 \\
\langle sc_2(\sigma_M(x), \text{Eq}(x), \pi), \sigma_1 \rangle & \xrightarrow{o} \sigma_2 \quad \text{check_err}(\sigma, \sigma_1, \sigma_2, \pi) = 1 \\
\text{Eq, } \pi, \sigma_M & \vdash \sigma \xrightarrow{i} \sigma_M[x \mapsto \sigma_1(x), \ h \mapsto \sigma_1(h)]
\end{align*}
\]

\[
\begin{align*}
\langle sc_1(\sigma_M(x), \text{Eq}(x), \pi), \sigma \rangle & \xrightarrow{o} \sigma_1 \\
\langle sc_2(\sigma_M(x), \text{Eq}(x), \pi), \sigma_1 \rangle & \xrightarrow{o} \sigma_2 \quad \text{check_err}(\sigma, \sigma_1, \sigma_2, \pi) = 0 \\
\text{Eq, } \pi, \sigma_M & \vdash \sigma \xrightarrow{i} \sigma'[h \mapsto \max(\pi(h_{\text{min}}), \frac{\sigma(h)}{2})]
\end{align*}
\]

\[
\text{Eq, } \pi, \sigma_M \vdash \sigma \xrightarrow{i} \sigma'
\]

\textbf{Note:} \( sc_i \) represents the local expansion of equations.

Solving ODEs leads to a time discretization

Equations of the form \( \dot{x}_i = l_j \) are substituted by the \textbf{sequence of the equations} needed to realize the numerical integration.
Validation of the integration step

For adaptive step-size method: for all continuous state variables

\[
\text{integration error} = \frac{h_n \| x_{n+1} - y_{n+1} \|_\infty}{\max \left( \max \left( |x_{n+1}|, |x_n| \right), \frac{atol}{rtol} \right)} \quad \text{valid if} \quad \leq \text{rtol}.
\]

Strategy:

- **Success:** go the the zero-crossing detection step.
- **Failure:** reduce the step-size \( h_n \) in general only a division by 2. and restart the integration step with the new step-size.

Remark

The reduction of the step-size is done until the \( h_{\text{min}} \) is reached. In that case a simulation error may happen.
Zero-crossing event detection (non adaptive version)

Main steps

- **Detection** of zero-crossing event
  Is one of the zero-crossing changed its sign between \([t_n, t_n + h_n]\)?

- **Localization**: if detection is true
  Bracket the most recent zero-crossing time using bisection method.

- **Pass through** the zero-crossing event in two steps:
  - Set the next major output to the left bound of the bracket time.
  - Reset the solver with the state estimate at the right bound of bracket time.

Ingredients for the localization – 1

**Linear approximation** of the dynamic.

\[
\text{computeTz}(\sigma_L, \sigma_R) = \min_L \left\{ \sigma_L(t) - x_L \cdot \frac{\sigma_R(t) - \sigma_L(t)}{x_R - x_L} : \forall x \right\}
\]

with \(\min_L(t) = \min\{t_i \mid t_i \geq \sigma_L(t)\}\).

**Remark**: \(\sigma_L\) and \(\sigma_R\) represents the environments at the time enclosing the event.
Zero-crossing event detection (non adaptive version)

Main steps

- **Detection** of zero-crossing event
  Is one of the zero-crossing changed its sign between \([t_n, t_n + h_n]\)?

- **Localization**: if detection is true
  Bracket the most recent zero-crossing time using bisection method.

- **Pass through** the zero-crossing event in two steps:
  - Set the next major output to the left bound of the bracket time.
  - Reset the solver with the state estimate at the right bound of bracket time.

Ingredients for the localization – 2

**Continuous extension** (method dependent) to easily estimate state.
For example, ode23

\[
\text{interpolX}(\sigma_{t_n}, \sigma_{t_n+h_n}, t) = (2\tau^3 - 3\tau^2 + 1)\sigma_{t_n}(x) + (\tau^3 - 2\tau^2 + \tau)(t_2 - t_1)p_1 \\
+ (-2\tau^3 + 3\tau^2)\sigma_{t_n+h_n}(x) + (\tau^3 - \tau^2)(t_2 - t_1)p_2
\]

with \(\tau = \frac{t-t_n}{h_n}\)
Update integration step-size

Goals of this step

Performance issue:
- increase the step-size to reduce the number of simulation loop.
- or decrease the step-size to increase the accuracy.

Scheduling issue: must not missed a discrete sampling time.

Main rule

\[ h' = \text{changeH}(\sigma(h), \pi) \quad t' = \min\{\tau \in \text{Eq(sampling)} : \tau > \sigma(t)\} \]

\[ \text{Eq, } \pi, \sigma_M \vdash \sigma \xrightarrow{h} \sigma[h \mapsto \min(h', t' - \sigma(t))] \]

Note: changeH increase or decrease \( h \) in function of the integration error.

Remark

If a zero-crossing occurred the step-size is left unchanged in regards to the performance issue.
Conclusion
Conclusion and future work

Conclusion
Despite Simulink is a closed source software, the main algorithms in the solver are known.

Hope
The definition of a formal semantics will help increasing the confidence in the Simulink design flow with new verification methods.

Future work
- Pursuing the formalization of the semantics: reset state, adaptive zero-crossing, integration method like ode113, etc.
- Development of a homemade simulator of Simulink models to validate experimentally our semantics.
- Development of a set-based simulator in order to address the reachability problem of Simulink models.