Non-Standard Semantics of Hybrid Systems Modelers

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Trends in Hybrid Systems Modelers

- From Simulink/Stateflow — data-flow/block-diagram programming:
  - hybrid systems, mode switching
  - input/output oriented
  - ODEs (Ordinary Differential Equations)

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  - Bond Graphs
  - DAEs (Differential Algebraic Equations)

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- To Modelica or SimScape — component based modeling:
  - Modeling from first principles
  - No inputs, no outputs, but constraints
  - Bond Graphs
  - DAEs (Differential Algebraic Equations)
- Some invariants, though:
  - different tools yield different executions for the same model
  - problems in handling zero-crossings and resets
  - unwanted interactions for seemingly non-interacting sub-systems

Why do these difficulties remain?

- Hybrid systems modelers semantics remains problematic
  Much more so than for synchronous languages
- Cascades of zero-crossings and mode changes cause difficulties
- Causality analysis is difficult

$\Longrightarrow$

- Scheduling cascaded reset actions and mode changes is problematic
- Step adaptation policies may cause unwanted interactions for seemingly non-interacting sub-systems

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Some contradictory requirements regarding semantics:

1. The semantic function must be statically defineable
2. Computers can only run according to discrete steps
   $\Longrightarrow$ Discretization must be part of the semantic map
3. Adaptive discretization schemes are chosen at run-time

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Slicing, in a sound way, the system into its discrete and continuous parts:

- the discrete part should include cascaded zero-crossings and mode changes and can be handled according to discrete time systems techniques
- the continuous part can be handled as in numerical maths, using proper adaptive discretization schemes

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Some further difficulties with existing semantic approaches:

- the semantics of a system must say what its behaviors are
  - this is usually expressed in terms of sets of behaviors
  - or alternatively in terms of a “simple machine” that can produce them (e.g., SOS rules)

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  - the ODE has no unique solution; no solution;
  - the hybrid system is zeno; is very zeno;

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  - the ODE has no unique solution; no solution;
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  this is gently ignored. . .
- well, let’s put effort in stating assumptions ensuring well defined semantics; are these stable under system composition?

Objectives of our research

Accept/Reject programs based on sound and simple criteria

- have a clean semantics

Keep continuous time gently separated from discrete time

- ensure that continuous/discrete can be separated at compile time (typing)

Reuse off-the-shelf engines

- slicing a system into its discrete/continuous parts
- submiting the continuous part to an off-the-shelf solver
- submiting the discrete part to an off-the-shelf synchronous language engine

Fix the problems with the unwanted interactions

Help improving physical system modeling, where hybrid systems can involve DAEs

Introduction and Objectives

Some examples

Non-Standard Hybrid Systems (for the math-averse)

Non-Standard Analysis and Standardisation (for the fan)

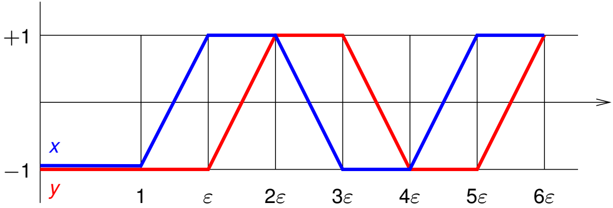
Non-Standard Hybrid Systems and their Standardisation

Conclusion and Main Messages

Some examples 1: infinite cascade

$$\left\{ \begin{array}{llll} \dot{y} = & 0 \textbf{ init } -1 & \textbf{reset } [1, -1] & \textbf{every up}[x, -x] \\ \dot{x} = & 0 \textbf{ init } -1 & \textbf{reset } [-1, 1, 1] & \textbf{every up}[y, -y, z] \\ \dot{z} = & 1 \textbf{ init } -1 & & \end{array} \right.$$

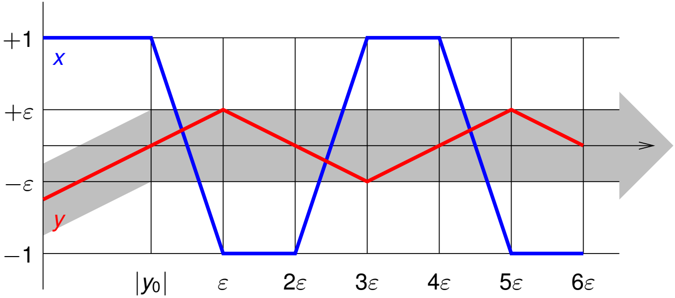
Note that z is just a physical clock. So, such an example can arise with "discrete" systems following the discrete/hybrid classification in force in the community of hybrid systems modelers.



here and subsequently, ε is infinitesimal

Some examples 2: sliding mode

$$\begin{cases} \dot{x} = 0 \textbf{ init } -\mathrm{sgn}(y_0) \textbf{ reset } [-1, 1] \textbf{ every up}[y, -y] \\ \dot{y} = x \textbf{ init } y_0 \end{cases}$$

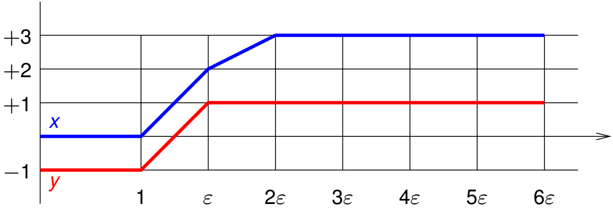


This is a simple form for an ABS system. Corresponding "averaged" system is:

$$\dot{\mathbf{y}} = \begin{cases} -\mathrm{sgn}(y_0), & \text{for the interval } [0, |y_0|\,) \\ 0 & \text{for } [\,|y_0|, \infty), \end{cases}$$

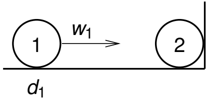
Some examples 3: finite cascade

$$\begin{cases} \dot{x} = 0 \textbf{ init } \quad 0 \textbf{ reset } [\textbf{last}\,(x) + 1, \textbf{last}\,(x) + 2] \textbf{ every up}[y, z] \\ \dot{z} = 1 \textbf{ init } -1 \\ \dot{y} = 0 \textbf{ init } -1 \textbf{ reset } [1] \textbf{ every up}[z] \end{cases}$$



Here the question is: how should the reset on x and y be performed? Here we have adopted a mirco-step interpretation reflecting causality between the two resets. A different interpretation is often proposed by existing modelers.

Some examples 4: balls on wall



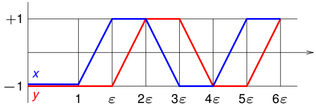
$$\left\{\begin{array}{lcl} \dot{x}_1 & = & v_1 \textbf{ init } d_1 \\ \dot{x}_2 & = & v_2 \textbf{ init } d_2 \\ \dot{v}_1 & = & 0 \textbf{ init } w_1 \textbf{ reset last}\,(v_2) \textbf{ every up}[x_1 - x_2] \\ \dot{v}_2 & = & 0 \textbf{ init } w_2 \textbf{ reset}\,[\textbf{last}\,(v_1)\,, -\textbf{last}\,(v_2)] \textbf{ every up}[x_1 - x_2, x_2] \end{array}\right.$$

Here the difficulty is the cascade involving

1. ball 1 hitting ball 2, resulting in ball 2 moving to the right (reset)
2. which causes ball 2 to hit the wall immediately (ODE activated for zero time)
3. resulting in ball 2 moving backward (reset)
4. followed by the symmetric sheme.

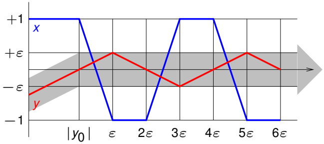
Questions

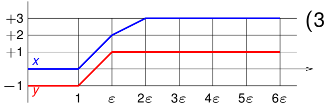
- Can we propose a semantic domain for these (and all) examples?
- Can we use it
  - to identify example (1) as pathological, but not example (2)?
  - to decide on the semantics of example (3)?
  - to give a semantics to example (4)?
- More generally, can we develop a semantic domain to serve as a mathematical basis for the management of (possibly cascaded) zero-crossings?



(1)

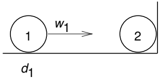
(2)





(3)

(4)



The great idea: non-standard analysis

Suppose for a while that we can give a formal meaning to the following:

$$\dot{y} = x \quad \text{means, by definition:} \quad \frac{y_{t+\partial} - y_t}{\partial} = x_t$$

where ∂ is infinitesimal

Let's make a trial use of non-standard anaysis. The ε of our examples will be identified with the above ∂. By doing so, our drawings become the semantics of cascades and ODEs' semantics is written as transition relations involving ∂.

Non-Standard Time Base

Fix an infinitesimal base step ∂

$$\text{time base} \; : \; \mathbb{T} \quad = \quad \{t_n = n\partial \mid n \in {}^\star\mathbb{Z}\}$$

$$\begin{aligned} \text{define } \forall t \in \mathbb{T} \; : \; {}^\bullet t &= \max\{s \mid s \in \mathbb{T}, s < t\} \\ t^\bullet &= \min\{s \mid s \in \mathbb{T}, s > t\} \end{aligned}$$

$\mathbb{T}$ offers *“le beurre et l’argent du beurre”* (popular french idiom):

- (i) $\mathbb{T}$ is totally ordered
- (ii) every subset of $\mathbb{T}$ that is bounded from above by a finite (non-standard) number has a unique maximal element
- (iii) $\mathbb{T}$ is dense in $\mathbb{R}$

By (i) and (ii) $\mathbb{T}$ looks “discrete”; by (iii), $\mathbb{T}$ looks “continuous”

Non-Standard Time Base

$$\begin{array}{rcl} \mathbb{T} & = & \{t_n = n\partial \mid n \in {}^\star\mathbb{Z}\} \\ \forall t \in \mathbb{T} \,:\, {}^\bullet t & = & \max\{s \mid s \in \mathbb{T}, s < t\} \\ t^\bullet & = & \min\{s \mid s \in \mathbb{T}, s > t\} \end{array}$$

ODE:

$$\underbrace{\dot{x} = f(x,u)}_{\text{(possibly not well defined)}} \quad \longleftrightarrow \quad \underbrace{x_t = x_{{}^\bullet t} + \partial \times f(x_{{}^\bullet t}, u_{{}^\bullet t})}_{\text{(always well defined)}}$$

Streams of events generated by the zero-crossings of x:

$$\begin{array}{lll} \zeta_x & =_{\text{def}} & \{t \in \mathbb{T} \mid x_{{}^\bullet t} < 0 \wedge x_t \geq 0\} \quad \text{(always well defined)} \\ & \approx & \left\{s \in \mathbb{R} \mid x_{s\_} < 0 \wedge x_s \geq 0\right\} \quad \text{(possibly not well defined)} \end{array}$$

Cascades following t:

$$t,\ {}^\bullet t,\ {}^{\bullet\bullet}t,\ {}^{\bullet\bullet\bullet}t, \ldots \quad \longleftrightarrow \quad ????$$

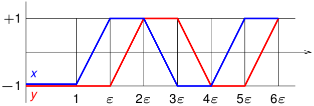
No standard counterpart using $\mathbb{R}$; $\mathbb{R} \times \mathbb{N}$ sufficient for finite cascades ("super-dense" time). Some cascades are worse (example 1) and cannot find their semantics in super-dense time

Back to the examples

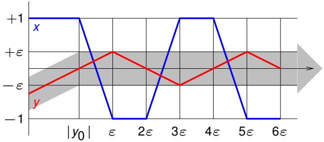
Can we propose a semantic domain for these (and all) examples?
The drawings show the non-standard semantics with $\partial := \varepsilon$

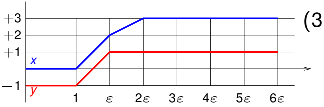
Can we use it — yes we can

- to identify example (1) as pathological? — easy
- to identify example (2) as non-pathological? — less easy
- to decide on the semantics of example (3)? — easy
- to give a semantics to example (4)? — subtle



(1) (2)





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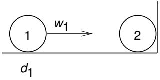

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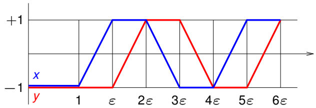
Can we use it yes we can

- to identify example (1) as pathological? easy

The figure shows the non-standard semantics. The system oscillates for the whole $\mathbb{T}$ ("for ever"), for a non-standard number of times. Note that the sequence of instants $n\varepsilon$ tends to infinity because n can itself be an infinite non-standard integer. This trajectory possesses no standardisation.



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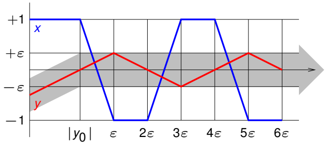
Can we use it — yes we can

- to identify example (2) as non-pathological? — less easy

The figure shows the non-standard semantics. The system oscillates for the whole $\mathbb{T}$ (“for ever”), for a non-standard number of times. However, while the blue trajectory oscillates between -1 and $+1$, the red one oscillates between $-\varepsilon$ and $+\varepsilon$, and it can be proved that the standard part of this trajectory is indeed the thick grey polyline in which ε is intepreted as zero.



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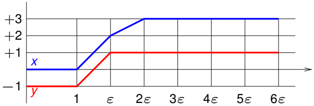
- to decide on the semantics of example (3)? — easy

The figure shows the non-standard semantics. The system has a first zero-crossing at $t = 1$, which causes a second one to occur on the blue trajectory at $t = 1 + \varepsilon$. This yields a classical super-dense time semantics.



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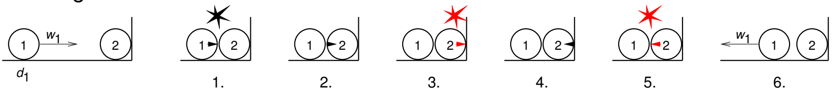
Non-standard semantics of the colliding balls example:

1. $t = \partial$, $x_1 = \partial \cdot w_1 > 0 \Rightarrow$ z-c (zero-crossing) on $x_1 - x_2$.
2. $\Rightarrow$ at $t = 2\partial$ balls exchange velocities: $v_1 = 0$ and $v_2 = w_1$.
3. $t = 3\partial$, $x_1 = 2\partial \cdot w_1$ and $x_2 = \partial \cdot w_1 \Rightarrow$ ODE has immediate z-c on x_2
4. $t = 4\partial$, $x_1 = x_2 = 2\partial \cdot w_1$, $v_1 = 0$ and $v_2 = -w_1$.
5. $t = 5\partial$, $x_1 = 2\partial \cdot w_1$ and $x_2 = \partial \cdot w_1 \Rightarrow$ z-c $x_1 - x_2$
6. $\Rightarrow$ at $t = 6\partial$, $x_1 = 2\partial \cdot w_1$, $x_2 = 0$, $v_1 = -w_1$ and $v_2 = 0$.

Then, ball 1 moves toward $-\infty$ according to the ODEs and no further zero-crossings occur.



What is needed to establish the above on firm bases?

Two things are needed:

1. To establish on firm bases the juggling we plaid with ε and ∂ without care for both continuous and discrete dynamics
2. To relate it to “normal life semantics” where discrete dynamics, continuous dynamics and hybrid dynamics may or may not be well defined (existence/uniqueness/nonzenoness of solutions), not to speak about composition thereof

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Answers to the above will be

1. Non-Standard analysis *seriously*; don’t be afraid. . .
2. Standardisation of non-standard entities

Let’s go, then!

What is needed to establish the above on firm bases?

- What Non-Standard semantics yields:
  1. NS semantics is always defined; it involves dynamical systems indexed by $\mathbb{T} = \partial\times$ "as if they were discrete"

     $$\text{hybrid system program} \;\rightarrow_\partial\; \text{NS semantics}$$

  2. Systems always compose

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  2. Systems always compose

- Standardisation principle: There exists a standardisation map

  $$\text{hybrid system program} \;\rightarrow_{\partial}\; \text{NS semantics} \;\mapsto\; \text{S semantics}$$

  such that

  1. it is a partial map (sometimes NS systems have no S counterpart)
  2. when standardisation exists, then the above double map does not depend on ∂: NS semantics is intrinsic
  3. when system composition is well defined in the S domain, then we get commutative diagrams

Non-Standard Analysis

A bit of history

- Born in 1961 from Abraham Robinson, then developed by a small community of mathematicians.
- Proposed as a conservative enhancement of Zermelo-Fränkel set theory; some fancy axioms and principles; nice for the adicts
- Subject of controversies: what does it do for you that you cannot do using our brave analysis with $\forall\varepsilon\exists\eta\ldots$?
- 1988: a nice presentation of the topic by T. Lindstrom, kind of "*non-standard analysis for the axiom-averse*"
- 2006: used in Simon Bliudze PhD where he proposes the counterpart of a "Turing machine" for hybrid systems (supervised by D. Krob)

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Why is non-standard analysis interesting for the computer scientist?

- it offers a step-based view of continuous and hybrid systems
- it is non-effective; still, it is amenable to symbolic executions and can thus be used for symbolic analyses at compile (and even run) time

Non-Standard Analysis

The aim

- to augment $\mathbb{R} \cup \{\pm\infty\}$ with elements that are infinitely close to x for each $x \in \mathbb{R}$, call ${}^\star\mathbb{R}$ the result;
- ${}^\star\mathbb{R}$ should obey the same algebra as $\mathbb{R}$: total order, $+, \times, \ldots$ any $f : \mathbb{R} \mapsto \mathbb{R}$ extends to ${}^\star f : {}^\star\mathbb{R} \mapsto {}^\star\mathbb{R}$, etc

Idea:

- mimic the construction of $\mathbb{R}$ from $\mathbb{Q}$ as Cauchy sequences; candidates for infinitesimals include:

$$\text{close to } 0 \quad : \quad \left\{\frac{1}{\sqrt{n}}\right\} > \left\{\frac{1}{n}\right\} > \left\{\frac{1}{n^2}\right\} > 0$$

$$\text{close to } +\infty \quad : \quad \{\sqrt{n}\} < \{n\} < \{n^2\}$$

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Are we done? Not quite so:

- Sequences of reals $\{x_n\}$ generally do not converge
- Two sequences $\{x_n\}$ and $\{y_n\}$ converging to 0 may be s.t. $\{n \mid x_n > y_n\}$, $\{n \mid x_n < y_n\}$, and $\{n \mid x_n = y_n\}$ are all infinite sets

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Lindström: partition subsets of $\mathbb{N}$ into neglectible/non-neglectible ones, so that:

- finite or empty subsets are all neglectible
- neglectible sets are stable under finite unions
- for any subset P, either P or its complement is non-neglectible

Having such a decision mechanism relies on Zorn Lemma ($\approx$ axiom of choice) and is formalized as explained next.

Non-Standard Analysis: the idea of Lindstrom

Pick $\mathcal{F}$ a free ultrafilter of $\mathbb{N}$:

- $\emptyset \notin \mathcal{F}$, $\mathcal{F}$ stable by intersection
- $P \in \mathcal{F}$ and $P \subseteq Q$ implies $Q \in \mathcal{F}$
- P finite implies $P \notin \mathcal{F}$
- either P or $\mathbb{N} - P$ belongs to $\mathcal{F}$

Existence of $\mathcal{F}$ follows from Zorn's lemma ($\Leftrightarrow$ axiom of choice)

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Define:

$$\mu(P) = \text{ if } P \in \mathcal{F} \text{ then 1 else 0}$$

- $P \cap Q = \emptyset \Rightarrow \mu(P \cup Q) = \mu(P) + \mu(Q)$; $\mu(\mathbb{N}) = 1$
- P finite implies $\mu(P) = 0$: P is neglectible

Non-Standard Analysis: the idea of Lindstrom

$(x_n), (x'_n) \in \mathbb{R}^{\mathbb{N}}$, define $(x_n) \approx (x'_n)$ iff set $\{n \mid x_n \neq x'_n\}$ is neglectible

$$^\star\mathbb{R} = \mathbb{R}^{\mathbb{N}} / \approx \quad ; \quad \text{elements of } {}^\star\mathbb{R} \text{ are written } [x_n]$$

- For any two $(x_n), (y_n)$ exactly one among the sets $\{n \mid x_n > y_n\}, \{n \mid x_n < y_n\}, \{n \mid x_n = y_n\}$, is non-neglectible
  $\Longrightarrow$
  any two sequences can always be compared modulo $\approx$

- By pointwise extension, a 1^{st}-order formula is true over ${}^\star\mathbb{R}$ iff it is true over $\mathbb{R}$: this is known as the transfer principle
- Say that

$$x = st([x_n]) \quad \text{if} \quad x_n \to x \quad \text{modulo neglectible sets}$$

Non-Standard Analysis: the idea of Lindstrom

Theorem: [standardisation] Any non-standard real $[x_n]$ possesses a unique standard part

Proof:

1. Pick
$$x = \sup\{u \in \mathbb{R} \mid [u] \leq [x_n]\}$$
where $[u]$ denotes the constant sequence equal to u.
2. Since $[x_n]$ is finite, x exists; remains to show that $[x_n] - x$ is infinitesimal.
3. If this is not true,
   - then there exists $y \in \mathbb{R}, y > 0$ such that $y < |x - [x_n]|$,
   - that is, either $x < [x_n] - [y]$ or $x > [x_n] + [y]$,
   - which both contradict the definition of x.
4. The uniqueness of x is clear, thus we can define $st([x_n]) = x$.

Infinite non-standard reals have no standard part in $\mathbb{R}$.

Integrals, ODE, and the Standardisation Principle

- internal functions and sets by pointwise extension:

$$\forall n, g_n : \mathbb{R} \mapsto \mathbb{R} \text{ yields } [g_n] : {}^\star\mathbb{R} \mapsto {}^\star\mathbb{R} \text{ by } [g_n]([x_n]) = [g_n(x_n)]$$

- Pick ∂ infinitesimal and $N \in {}^\star\mathbb{N}$ such that $(N-1)\partial < 1 \leq N\partial$, and consider the set
$$T = \{0, \partial, 2\partial, \ldots, (N-1)\partial, 1\}$$
By definition, if $\partial = [d_n]$, then $N = [N_n]$ with $N_n = \frac{1}{d_n}$ and $T = [T_n]$ with
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- For $f : [0,1] \mapsto \mathbb{R}$ a continuous function and ${}^\star f = [f, f, \ldots]$ its non-standard version

$$\left[\sum_{t \in T_n} \frac{1}{N_n} f(t_n)\right] = \sum_{t \in T} \frac{1}{N} {}^\star f(t)$$

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we claim this

Integrals, ODE, and the Standardisation Principle

Theorem: [standardisation] if $f : [0,1] \to \mathbb{R}$ is continuous, then

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Proof: If $f : \mathbb{R} \to \mathbb{R}$ is a standard function, we always have

$$\sum_{t\in T} \frac{1}{N} {}^\star f(t) = \left[\sum_{t\in T_n} \frac{1}{N_n} f(t_n)\right] \tag{1}$$

Now, f continuous implies $\sum_{t\in T_n} \frac{1}{N_n} f(t_n) \to \int_0^1 f(t)$, so, by definition of non-standard reals,

$$\int_0^1 f(t) = st\left(\sum_{t\in T} \frac{1}{N} {}^\star f(t)\right) \tag{2}$$

Integrals, ODE, and the Standardisation Principle

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- Thus, if f is smooth so that its Riemann integral is well defined, then any non-standard formulation of the integral of f has $\int_0^1 f(t)$ as its standard part
- The same philosophy applies to ODEs and Hybrid Systems

Integrals, ODE, and the Standardisation Principle

For every $0 < t \leq 1$:

$$\int_0^t f(u)du = st\left(\sum_{u \in T, u \leq t} \frac{1}{N} {}^\star f(t)\right) \quad \text{(Non-standard Riemann integral)}$$

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Set $\partial = \frac{1}{N}$ and consider the ODE $\dot{x} = f(x,t), x_0$, in integral form

$$\begin{aligned} x(t) &= x_0 + \int_0^t f(x(u),u)du \quad \text{(with the needed smoothness)} \\ x(t) &= st\left(x_0 + \sum_{k\,:\,0\leq k\partial\leq t} \frac{1}{N}{}^\star f({}^\star x(k\partial), k\partial)\right) \end{aligned}$$

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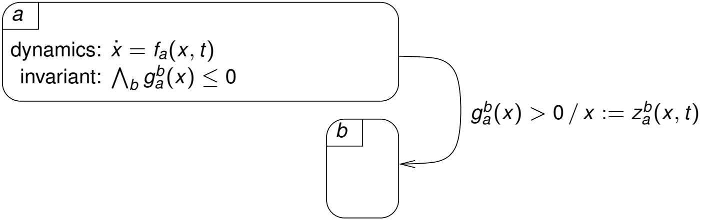
where ${}^{\star}x$ is the non-standard semantics of the above ODE with time basis ∂:

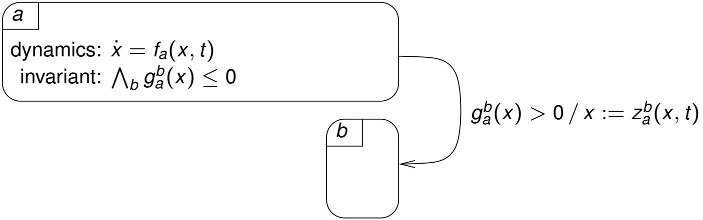
$$\begin{cases} {}^{\star}x(t_k) &= {}^{\star}x(t_{k-1}) + \partial \times f({}^{\star}x(t_{k-1}), t_{k-1}) \\ {}^{\star}x(t_0) &= x_0 \end{cases} \tag{4}$$

Theorem: [standardisation]
(4) is always defined as a non-standard dynamical system
(3) only holds if the ODE has a solution

Non-Standard Hybrid Systems, Standardisation Principle



Non-Standard Hybrid Systems, Standardisation Principle



Standard semantics:

- spending standard > 0 duration within modes: ODE
- finite cascades of mode changes: super-dense time $(t, n) \in \mathbb{R} \times \mathbb{N}$

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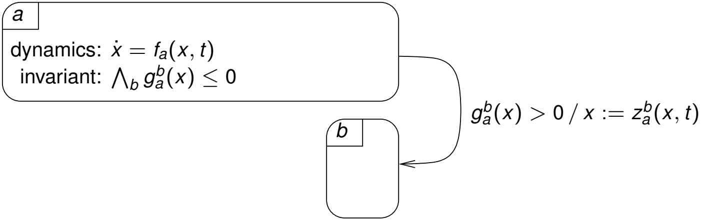

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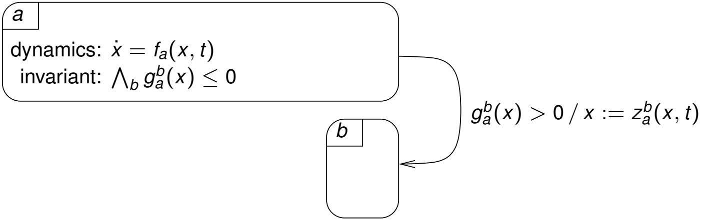

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- spending $\geq$0 duration within modes: non-standard ODE
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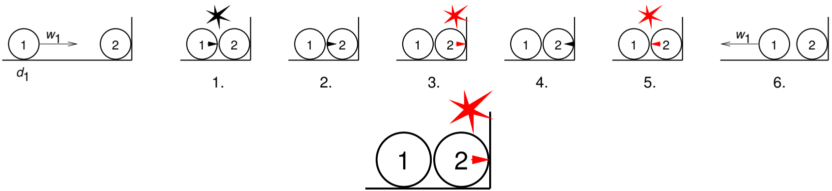
Theorem: [standardisation] if the S semantics is well-defined, then it is the standardisation of the NS (∂-dependent) semantics, for any choice of ∂

Non-Standard Hybrid Systems, Standardisation Principle



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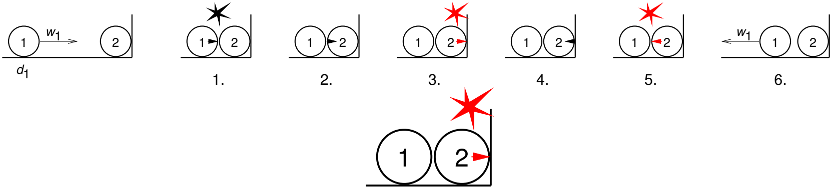
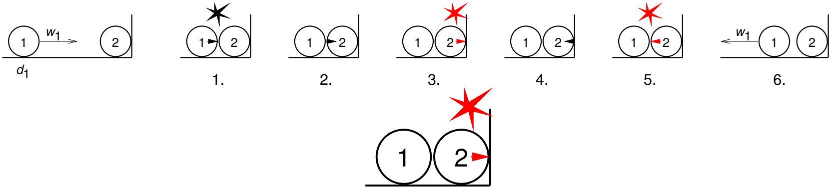


Non-standard symbolic simulation of the colliding balls example:

1. $t = \partial$, $x_1 = \partial \cdot w_1 > 0 \Rightarrow$ z-c (zero-crossing) on $x_1 - x_2$.
2. $\Rightarrow$ at $t = 2\partial$ balls exchange velocities: $v_1 = 0$ and $v_2 = w_1$.
3. $t = 3\partial$, $x_1 = 2\partial \cdot w_1$ and $x_2 = \partial \cdot w_1 \Rightarrow$ ODE has immediate z-c on x_2
4. $t = 4\partial$, $x_1 = x_2 = 2\partial \cdot w_1$, $v_1 = 0$ and $v_2 = -w_1$.
5. $t = 5\partial$, $x_1 = 2\partial \cdot w_1$ and $x_2 = \partial \cdot w_1 \Rightarrow$ z-c $x_1 - x_2$
6. $\Rightarrow$ at $t = 6\partial$, $x_1 = 2\partial \cdot w_1$, $x_2 = 0$, $v_1 = -w_1$ and $v_2 = 0$.

Non-Standard Hybrid Systems, Standardisation Principle



In this example, we successively have, within an infinitesimal period of time:

1. a first cascade of z-c
2. the launching of an ODE with an immediate z-c
3. another cascade of z-c, followed by the symmetric scheme.

Provided that such a cascade of {z-c + ODE micro-steps} remains finite, a super-dense time semantics can be given. Execution is by executing the symbolic non-standard semantics: Extended Standardisation Principle.

Conclusion and main messages

Non-Standard Semantics is not just for the adicts, it's useful:

- it provides a semantic domain in which
  - all hybrid systems possess a semantics; and,
  - when the ordinary semantics is well defined, then it is the standardisation of the non-standard semantics
- it provides a crisp understanding of cascades of zero-crossings
- it helps deciding between rejecting/accepting hybrid systems based on their semantics
- non-standard symbolic simulation is useful and effective (balls-on-wall); remains to be well understood and implemented...

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Of course, most important are tools and non-standard semantics is not a panacea, after all